Class 7, given on Jan 20, 2010, for Math 13, Winter 2010

1. One more example of iterated integrals

There may be situations where you will be required to split an iterated integral into several pieces because the functions describing the boundary of the region of integration might be defined in a piecewise manner. Consider the somewhat strange region D defined by $-1 \leq x \leq 1, 0 \leq y \leq g(x)$, where $g(x) = x + 1$ if $-1 \leq x \leq 0$, and $g(x) = 1 - x^2$ if $0 \leq x \leq 1$. If we want to evaluate the double integral of $f(x, y)$ over D using an iterated integral, we would write

$$\int_{-1}^{1} \int_{0}^{g(x)} f(x, y)\, dy\, dx.$$

However, this description of the bounds of the inner integral is not particularly useful from a computational perspective. We instead break up the region of integration into two parts, corresponding to $-1 \leq x \leq 0$ and $0 \leq x \leq 1$, so that we can describe the upper boundary of D using familiar functions:

$$\int_{-1}^{0} \int_{0}^{x+1} f(x, y)\, dy\, dx + \int_{0}^{1} \int_{0}^{1-x^2} f(x, y)\, dy\, dx.$$

This is an illustration of the general fact that if D_1, D_2 are two regions of the xy plane which do not overlap except possibly at their boundaries, then

$$\iint_{D_1 \cup D_2} f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA.$$

2. A review of polar coordinates

There are many situations where we may want to integrate a function over a circular or elliptical domain. In this situation, describing the upper and lower boundary of the domain involves functions like $\sqrt{1 - x^2}$, which may be hard to work with. Instead, we might instead ask if it is possible to calculate these integrals with respect to *polar coordinates*, which are a more natural coordinate system to use when dealing with circles, ellipses, and more elaborate curves, such as cycloids.

Before discussing integration with polar coordinates, we will review the concept of polar coordinates. Instead of describing the position of a point in the xy plane by using its distance from the x and y axes, we can describe a point by giving its distance from the origin, and the angle the segment connecting the point with the origin makes with the positive x-axis. This is somewhat akin to how points might be described in a radar system.

More specifically, if we say that a point in the xy-plane has polar coordinates (r, θ), then we are talking about the point (x, y) with rectangular coordinates

$$x = r \cos \theta, y = r \sin \theta.$$

One complication that arises when using polar coordinates is the fact that the polar coordinates for a given point are not unique. For example, consider the point with polar coordinates $(1, \pi/2)$. This is the point $(0, 1)$ in rectangular coordinates. However, the points with polar coordinates $(-1, -\pi/2), (1, 5\pi/2), (1, 9\pi/2), \ldots$ all represent this point as well. In general, the point (r, θ) can also be represented by $(r, \theta + 2n\pi)$, where n is any integer – this represents the fact that we can go 2π radians around a circle and return to the same point, and also by $(-r, \theta + \pi)$, which represents the fact that going around a circle by π radians is the same as reflecting a point through the origin.

We want equations which let us convert from rectangular to polar coordinates and vice versa. Given a point (x, y) in rectangular coordinates, we can let $r = \sqrt{x^2 + y^2}$. This guarantees that r is always non-negative and also determines r uniquely. The formula for θ is not quite as nice. One possibility is

$$\theta = \arctan(y/x) \text{ if } x > 0, \theta = \arctan(y/x) + \pi \text{ if } x < 0$$

up to the ambiguity by a multiple of 2π. However, in general, if you need to convert rectangular to polar coordinates by hand, it is frequently easier to just draw a sketch of the point (x, y) and then calculate θ from trigonometric principles.

Converting from polar to rectangular coordinates is much easier. If a point has polar coordinates (r, θ), then its rectangular coordinates are given by the formulas $x = r\cos\theta, y = r\sin\theta$ listed above.

Examples.

- Polar coordinates are very well suited to working with circles. For example, a circle with equation $x^2 + y^2 = 1$ is defined by the simple polar equation $r = 1$.
- The equation $r = \theta$ defines a spiral, sometimes known as the *spiral of Archimedes*.
- The polar equation $\theta = \pi/2$ is the same as the line $y = 0$.

3. Integrating using polar coordinates

Suppose we want to calculate the volume of a sphere by taking a double integral. For example, consider the sphere $x^2 + y^2 + z^2 = 1$; to calculate this volume using an integral we will want to calculate an integral of the form

$$\iint_D \sqrt{1 - x^2 - y^2}\, dA$$

where D is the disc $x^2 + y^2 \leq 1$. (This integral is actually equal to half the volume of the unit sphere.) As an iterated integral, this double integral is equal to

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx.$$

It is possible to evaluate this integral, although it is not entirely pleasant. How can we use polar coordinates to simplify the calculation of this integral?

We should start by converting the function being integrated, which in this case is, $f(x, y) = \sqrt{1 - x^2 - y^2}$, to polar coordinates, by using the substitution $x = r\cos\theta, y = r\sin\theta$. We see that the result is

$$f(r\cos\theta, r\sin\theta) = \sqrt{1 - r^2}.$$

(We use the fact that $\cos^2\theta + \sin^2\theta = 1$, which is a basic identity to know when working with polar coordinates or performing trigonometric integrals.) We also need to convert the bounds of integration, which indicate the domain we are integrating over, from rectangular to polar coordinates. When doing this it is often best to draw a sketch.

In this example, we are integrating over the unit disc $x^2 + y^2 \leq 1$. This is evidently the same region as that defined by polar inequalities $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. Notice that we need to restrict θ so that every point of the disc, except possibly those which lie on the positive x-axis, is represented only once by polar coordinates.

It turns out that to calculate an integral using polar coordinates, not only do we need to convert the function being integrated from rectangular to polar coordinates as well as the region of integration, but we also need to insert an addition factor of r in the

integrand. More specifically, if f is continuous on a region D defined by polar coordinates $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

This extra factor of r might seem somewhat mysterious, but it is analogous to how when performing a u-substitution in integration, we have a factor of $u'(x)$ which appears:

$$\int f(u)\,du = \int f(u(x))u'(x)\,dx.$$

We are carrying out the higher dimensional analogue of u-substitution using the specific change of variables $x = r\cos\theta, y = r\sin\theta$, and the factor of r appearing in the integrand plays the role of $u'(x)$. We will see how this fits into a more general formula for change of variables in a few weeks.

In our example, converting the integral from rectangular to polar coordinates yields

$$\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r\,dr\,d\theta.$$

Notice that the extra factor of r makes the resulting integral easy to compute by u-substitution! We have

$$\int_0^{2\pi} \frac{(1-r^2)^{3/2}}{-2} \cdot \frac{2}{3}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{2\pi}{3}.$$

This accords with what we know to be true from geometry, which tells us that a sphere of radius 1 has volume $4\pi/3$.

Example. Use polar coordinates to evaluate the double integral

$$\iint_D 1 - \sqrt{x^2+y^2}\,dA$$

where D is the unit disc $x^2 + y^2 \le 1$. To the volume of what geometric figure does this double integral equal? Use this to check that your answer is correct.

We first draw a sketch of the geometric figure whose volume this double integral represents. The graph of $z = 1 - \sqrt{x^2+y^2}$ can be thought of as the graph of $z = 1 - r$, where we convert rectangular coordinates in the xy plane to polar coordinates. But this is just a cone whose base is the disc D and has height 1.

Again, notice that while it is possible to evaluate this integral in rectangular coordinates using trigonometric substitutions, it certainly seems like it is (and will be) easier to carry out this integral using polar coordinates. The unit disc D is given by polar inequalities $0 \le r \le 1, 0 \le \theta \le 2\pi$, and we have already seen that the integrand can be expressed as $1 - r$. Therefore, the iterated integral we want to calculate is

$$\int_0^{2\pi} \int_0^1 (1-r) r\,dr\,d\theta.$$

(Remember that additional factor of r!) This is easy to calculate; we get

$$\int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3}\Big|_0^1 \right) d\theta = \int_0^{2\pi} \frac{1}{6}\,d\theta = \frac{\pi}{3}.$$

This agrees with what geometry tells us, since a cone whose base has area π and has height 1 will have volume $\pi/3$.